Master Theorem

29/05/2019

$T(n) = aT(\frac{n}{b}) + f(n)$

i) $f(n) = O(n^{\log_b(a-\varepsilon)})$ με $\varepsilon > 0$ τότε $T(n) = \Theta(n^{\log_b a})$

ii) $f(n) = \Theta(n^{\log_b a} \log^k n)$ με $k \geq 0$ τότε $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

iii) $f(n) = \Omega(n^{\log_b(a+\varepsilon)})$ με $\varepsilon > 0$ τότε $T(n) = \Theta(f(n))$

Άσκηση 1

1) $T(n) = 4T(\frac{n}{2}) + n$

2) $T(n) = 4T(\frac{n}{2}) + n^2$

3) $T(n) = 4T(\frac{n}{2}) + n^3$

Λύση

1) $T(n) = 4T(\frac{n}{2}) + n$

$a = 4, \quad b = 2 \quad , f(n) = n$

$f(n) = n^{\log_2(4-2)} = O(n^{\log_2(4-2)})$

$T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$

2) $T(n) = 4T(\frac{n}{2}) + n^2 \quad , \quad a = 4 \ , \ b = 2$

$f(n) = n^2 = n^{\log_2 4} = \Theta(n^{\log_2 4})$

$T(n) = \Theta(n^{\log_2 4} \log n)$

3) $T(n) = 4T(\frac{n}{2}) + n^3 \quad , \quad a = 4 \ , \ b = 2$

$f(n) = n^3 = n^{\log_2 4 + 4} = \Omega(n^{\log_2 4 + 4})$

$T(n) = \Theta(n^3)$

Άσκηση 2

$T(n) = 3T\left(\frac{n}{2}\right) + n^2$

Λύση

$a = 3 \quad b = 2 \quad f(n) = n^2$

$n^2 = n^{\log_2 3 + 1} \qquad f(n) = \Omega(n^{\log_2 3 + 1})$

$T(n) = \Theta(n^2)$

Άσκηση 3

$T(n) = 16T\left(\frac{n}{4}\right) + n$

Λύση

$a = 16 \quad b = 4 \quad f(n) = n$

$n = n^{\log_4 16 - 12} \qquad f(n) = O(n)$

$T(n) = \Theta(n^{\log_4 16}) = \Theta(n^2)$

Άσκηση 4

1) $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{2}$

2) $T(n) = 4T\left(\frac{n}{2}\right) + \log n$

Λύση

1) $a = 3 \quad b = 3 \quad f(n) = \frac{n}{2}$

$f(n) = \Theta(n)$

$T(n) = \Theta(n \log n)$

2) $a = 4 \quad b = 2 \quad f(n) = \log n$

$f(n) = \log n \leq n$

Τότε $f(n) \leq n$ άρα $f(n) \in O(n) = O(n^{\log_2 4 - 1})$

$T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$

Θεωρούμε τη συνάρτηση $T(n)$. Ορίζουμε

$T(n) \in O(f(n))$: $\exists$ c και n_0 : $T(n) \leq c f(n)$ $\forall n \geq n_0$

$T(n) \in \Omega(f(n))$: $\exists$ c και n_0: $T(n) \geq c f(n)$ $\forall n \geq n_0$

$T(n) \in \Theta(f(n))$ αν είναι και $O(f(n))$ και $\Omega(f(n))$

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int x = 0                                   c1
  for (int i = 0; i < n; i++)               n·c2
      for (int j = 0; j < n; j++)           c3·Σ_{j=1}^{n} j = c3·n(n+1)/2
          x++;                              c4·n(n+1)/2
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$$T(n) = c_1 + n \cdot c_2 + (c_3 + c_4)\frac{n(n+1)}{2} = O(n^2)$$